Section 4.4

Fourier Series and Frequency Spectra

- The Fourier series provides us with an entirely new way to view signals.
- Instead of viewing a signal as having information distributed with respect to *time* (i.e., a function whose domain is time), we view a signal as having information distributed with respect to *frequency* (i.e., a function whose domain is frequency).
- This so called frequency-domain perspective is of fundamental importance in engineering.
- Many engineering problems can be solved *much more easily* using the frequency domain than the time domain.
- The Fourier series coefficients of a signal x provide a means to *quantify* how much information x has at different frequencies.
- The distribution of information in a signal over different frequencies is referred to as the *frequency spectrum* of the signal.

- To gain further insight into the role played by the Fourier series coefficients c_k in the context of the frequency spectrum of the signal x, it is helpful to write the Fourier series with the c_k expressed in *polar form* as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} |c_k| e^{j(k\omega_0 t + \arg c_k)}.$$

- Clearly, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\omega_0$ that has been *amplitude scaled* by a factor of $|c_k|$ and *time-shifted* by an amount that depends on $\arg c_k$.
- For a given k, the *larger* $|c_k|$ is, the larger is the amplitude of its corresponding complex sinusoid $e^{jk\omega_0 t}$, and therefore the *larger the contribution* the kth term (which is associated with frequency $k\omega_0$) will make to the overall summation.
- In this way, we can use $|c_k|$ as a *measure* of how much information a signal x has at the frequency $k\omega_0$.

- The Fourier series coefficients c_k are referred to as the frequency spectrum of x.
- The magnitudes $|c_k|$ of the Fourier series coefficients are referred to as the magnitude spectrum of x.
- The arguments $\arg c_k$ of the Fourier series coefficients are referred to as the phase spectrum of x.
- Normally, the spectrum of a signal is plotted against frequency $k\omega_0$ instead of k.
- Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, the frequency spectrum is *discrete* in the independent variable (i.e., frequency.(
- Due to the general appearance of frequency-spectrum plot (i.e., a number of vertical lines at various frequencies), we refer to such spectra as line spectra.

- Recall that, for a *real* signal x, the Fourier series coefficient sequence c satisfies

$$c_k = c^*_{-k}$$

)i.e., c is *conjugate symmetric*), which is equivalent to

$$|c_k| = |c_{-k}| \quad \text{and} \quad \arg c_k = -\arg c_{-k}.$$

- Since $|c_k| = |c_{-k}|$, the magnitude spectrum of a *real* signal is always *even*.
- Similarly, since $\arg c_k = -\arg c_{-k}$, the phase spectrum of a *real* signal is always *odd*.
- Due to the symmetry in the frequency spectra of real signals, we typically *ignore negative frequencies* when dealing with such signals.
- In the case of signals that are complex but not real, frequency spectra do not possess the above symmetry, and *negative frequencies become important*.

Section 4.5

Fourier Series and LTI Systems

- Recall that a LTI system H with impulse response h is such that $\mathcal{H}\{e^{st}\} = H(s)e^{st}$, where $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$. (That is, complex exponentials are *eigenfunctions* of LTI systems.)
- Since a complex sinusoid is a *special case* of a complex exponential, we can reuse the above result for the special case of complex sinusoids.
- For a LTI system H with impulse response h and a complex sinusoid $e^{j\omega t}$ (where ω is a real constant),

$$\mathcal{H}\{e^{j\omega t}\} = H(j\omega)e^{j\omega t},$$

where

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt.$$

- That is, $e^{j\omega t}$ is an *eigenfunction* of a LTI system and $H(j\omega)$ is the corresponding *eigenvalue*.
- We refer to $H(j\omega)$ as the frequency response of the system H.

- Consider a LTI system with input x, output y, and frequency response $H(j\omega)$.
- Suppose that the T-periodic input x is expressed as the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad \text{where } \omega_0 = 2\pi/T.$$

- Using our knowledge about the *eigenfunctions* of LTI systems, we can conclude

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0) e^{jk\omega_0 t}.$$

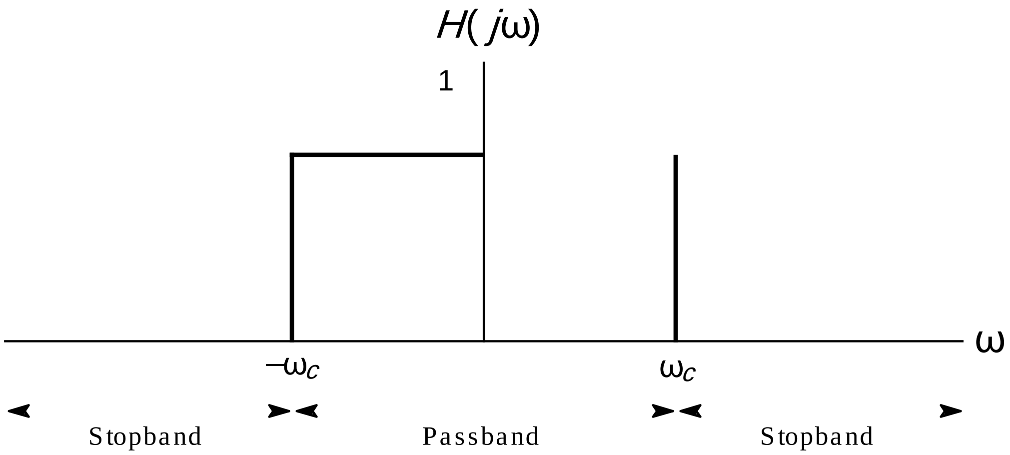
- Thus, if the input x to a LTI system is a Fourier series, the output y is also a Fourier series. More specifically, if $x(t) \overset{\text{CTFS}}{\longleftrightarrow} c_k$ then $y(t) \overset{\text{CTFS}}{\longleftrightarrow} H(jk\omega_0) c_k$.
- The above formula can be used to determine the output of a LTI system from its input in a way that *does not require convolution*.

- In many applications, we want to *modify the spectrum* of a signal by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a signal is called filtering.
- A system that performs a filtering operation is called a filter.
- Many types of filters exist.
- Frequency selective filters pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

- An ideal lowpass filter eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a *frequency response* of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \le \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

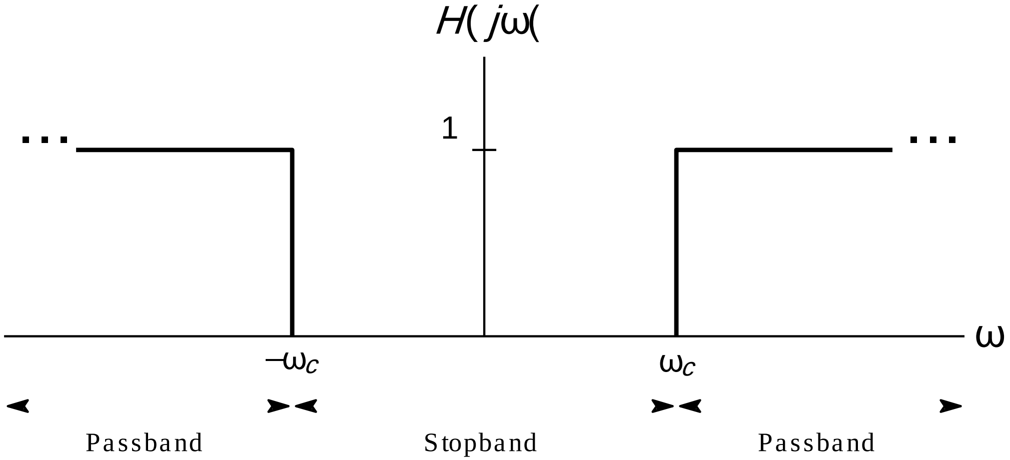
  where ω_c is the cutoff frequency.
- A plot of this frequency response is given below.



- An ideal highpass filter eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a *frequency response* of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

  where ω_c is the cutoff frequency.
- A plot of this frequency response is given below.

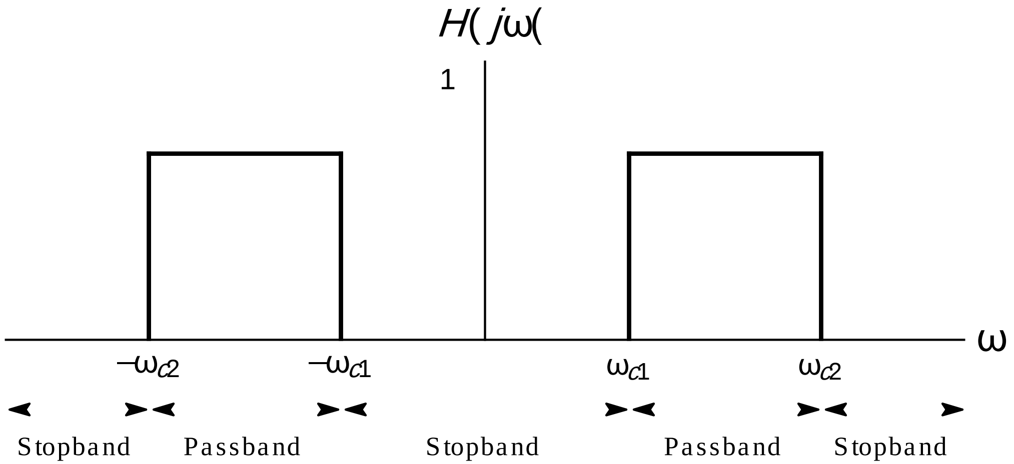


- An ideal bandpass filter eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.
- Such a filter has a *frequency response* of the form

$$H(j\omega) = \begin{cases} 1 & \text{for } \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

where the limits of the passband are ω_{c1} and ω_{c2}. A

- plot of this frequency response is given below.



Part 5

Continuous-Time Fourier Transform (CTFT(

- Fourier series provide an extremely useful representation for periodic signals.
- Often, however, we need to deal with signals that are not periodic. A
- more general tool than the Fourier series is needed in this case. The
- Fourier transform can be used to represent both periodic and aperiodic signals.
- Since the Fourier transform is essentially derived from Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.

Section 5.1

Fourier Transform